$\therefore \log[U+\sqrt{(U^2+\kappa)}]=a+\lambda,\ U+\sqrt{(U^2+\kappa)}=e^{a+\lambda}$.

Also, $\sqrt{(U^2+\kappa)}-U=\kappa e^{-(a+\lambda)}$, $2U=e^{a+\lambda}-\kappa e^{-(a+\lambda)}=Ce^{-a}+C'e^{a}$, where C, C', are constants. U not increasing indefinitely with a it follows that $C'=0$. When a is very small, (1) becomes

$$\underset{a\doteq 0}{L}\, U-\tfrac{1}{2}\pi=\underset{a\doteq 0}{L}\, -\int_0^\infty \frac{adx}{1+x^2}=\underset{a\doteq 0}{L}\, -\frac{a\pi}{2}=0;\ \therefore C=\tfrac{1}{2}\pi, \text{ and}$$

$$u=-\int_0^\infty \frac{x\sin ax dx}{1+x^2}=\frac{\pi}{2}e^{-a}\ (a \text{ being positive}).$$

But $\displaystyle\int_0^\infty \frac{x\sin ax}{1+x^2}dx=\frac{\pi}{2}-u=\frac{\pi}{2}(1-e^{-a})$.

Differentiating with respect to a,

$$\int_0^\infty \frac{x\cos ax}{1+x^2}dx=\frac{\pi}{2}e^{-a}.$$

$\therefore$ etc. (Cf. Roberts' *Treatise on the Integral Calculus*, Part I, p. 181.)

317. Proposed by C. N. SCHMALL, New York City.

A generating line of a right circular cylinder passes through the center of a sphere. The diameter of the cylinder is less than the radius of the sphere. Show that the surface of the cylinder included within the sphere is given by an elliptic integral.

Solution by A. M. HARDING, Fayetteville, Arkansas.

Let a=diameter of cylinder; r=radius of sphere. Choose the generating line of cylinder for z-axis. Let equation of sphere and cylinder be

$$x^2+y^2+z^2=r^2 \text{ and } x^2+y^2=ax,$$

respectively. Then

$$\frac{A}{4}=\int\int\left[1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}}dzdx.$$

Eliminate y and obtain $z^2+ax=r^2$. Hence z-limits are 0 and $\sqrt{(r^2-ax)}$, x-limits are 0 and a.

From equation of cylinder, we find

$$\frac{\partial y}{\partial x}=\frac{a-2x}{2y},\ \frac{\partial y}{\partial z}=0.$$

$$\therefore \frac{A}{4}=\int_0^a\int_0^{\sqrt{(r^2-ax)}}\left[1+\left(\frac{a-2x}{2y}\right)^2\right]^{\frac{1}{2}}dzdx=\int_0^a\int_0^{\sqrt{(r^2-ax)}}\frac{a}{2\sqrt{(ax-x^2)}}dzdx,$$

since $y^2=ax-x^2$.

$$\therefore A=2a\int_0^a\sqrt{\left(\frac{r^2-ax}{ax-x^2}\right)}dx.\quad \text{Putting } x=a\sin^2\phi,$$

$$A=2a\int_0^{\frac{1}{2}\pi}\frac{\sqrt{(r^2-a^2\sin^2\phi)}}{\sqrt{(a^3\sin^2\phi-a^2\sin^4\phi)}}.2a\sin\phi\cos\phi\, d\phi$$

$$=4a\int_0^{\frac{1}{2}\pi}\sqrt{(r^2-a^2\sin^2\phi)}.\frac{a\sin\phi\cos\phi}{a\sin\phi\sqrt{(1-\sin^2\phi)}}d\phi$$

$$=4ar\int_0^{\frac{1}{2}\pi}\sqrt{\left(1-\frac{a^2}{r^2}\sin^2\phi\right)}d\phi=4ar\,E\left(\frac{a}{r},\frac{\pi}{2}\right),\ a<r.$$

Also solved by Francis Rust and J. Scheffer.

318. Proposed by JOHN C. GREGG, Greencastle, Ind.

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

No correct solution of this problem has been received. Professor Feemster has given a solution finding the length of the thread. But the problem does not require that. Let us have a number of solutions of this problem. ED. F.

319. Proposed by C. N. SCHMALL, New York City.

Given $u=\dfrac{yz}{x}$, $v=\dfrac{zx}{y}$, $w=\dfrac{xy}{z}$, prove

$$\begin{vmatrix}\dfrac{\partial u}{\partial x}, & \dfrac{\partial u}{\partial y}, & \dfrac{\partial u}{\partial z}\\ \dfrac{\partial v}{\partial x}, & \dfrac{\partial v}{\partial y}, & \dfrac{\partial v}{\partial z}\\ \dfrac{\partial w}{\partial x}, & \dfrac{\partial w}{\partial y}, & \dfrac{\partial w}{\partial z}\end{vmatrix}=4xyz.$$

Solution by S. LEFSCHETZ, Ph. D., The University of Nebraska.

If $\triangle$ is the determinant proposed, we have: